The obstacle problem is a classic motivating example in the mathematical study of variational inequalities and free boundary problems. The problem is to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle. It is deeply related to the study of minimal surfaces and the capacity of a set in potential theory as well. Applications include the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control, and financial mathematics.

1. fluid filtration in porous media and heat transfer

The fluid flow around obstacles appears in diverse fields such as: aerodynamics (airflow around a building, a plane…) and hydraulics (superstructure: dams, bridges). Accompanied by the heat transfer, it concerns the energetic devices industry (heat pipes, electronic components, heat exchangers…). The presence of porous media is crucial for heat transfer enhancement [[1](https://link.springer.com/article/10.1007/s12648-018-1272-7#ref-CR1), [2](https://link.springer.com/article/10.1007/s12648-018-1272-7#ref-CR2)]. It satisfies the need of several domains as water filtration and catalytic converters equipment. Petroleum exploration and extraction are also concerned by this phenomenon. One of the most important applications of the heat transfer and fluid flow in porous media containing obstacles is the quenching of gas deflagrations.

1. optimal control

Scientists have investigated a lot of optimal control problems where the state is described by variational inequalities. These kind of problems have been extensively studied by many authors, as for example V. Barbu [3], F. Mignot and J.P. Puel [4], A. Friedman [5] or more recently Zheng-Xu He [6]. The optimal control problem as a “standard” control problem governed by a partial differential equation, involving state constraints which are not necessarily convex. Also Bergounioux[7] investigated optimal control problems governed by variational inequalities. and more precisely the obstacle problem. Since by adopting a numerical point of view, Bergounioux first relaxed the feasible domain of the problem ; then using both mathematical programming methods and penalization methods he got optimality conditions with smooth lagrange multipliers.

1. Financial applications

The parabolic obstacle problem refers to finding the smallest supersolution (for a given parabolic operator, and given domain and boundary data) over a given function (obstacle). This problem appears in applications, such as phase transitions (melting and crystallization) and American type contracts in finance. Of obvious reasons, the latter application has gained grounds in the recent past. Although many financial problems involve linear equations, there are many related problems with nonlinear governing equations, that require more delicate analysis. It is important to give an account of some new ideas and techniques, for nonlinear parabolic obstacle problem from free boundary regularity point of view. General results are exemplified in terms of applications in finance such as Petrosyan and Shahgholian[8] presented specific application of their algorithms for the valuation of the American put option of minimum of two underliers.

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